An Ecology and Economy Coupling Model. A global stationary state model for a sustainable economy in the Hamiltonian formalism

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ABSTRACT

The severity of the two deeply correlated crises, the environmental and the economic ones, needs to be faced also in theoretical terms; thus, the authors propose a model yielding a global "*stationary* state", following the idea of a "steady-state economics" by Georgescu-Rogen and Herman Daly, by constructing only one dynamical system of ecological and economic coupled variables. This is possible resorting to the generalized Volterra model, that, translated in the Hamiltonian formalism and its Hamilton equations, makes possible to "conjugate" every pair of variables, one economic, the other one ecological, in describing the behavior in time of a unique dynamical system.

Applying the model to two of the most relevant ecological-economic pairs of variables leads to a suggestive geometry in the "phase space" of the model: *the trajectories are curves wrapping a "donut", their set is the "stationary state" we were looking for*. Those trajectories are "*quasi-periodic motions*", characterized by *two frequencies*, for whose values a good estimate is provided in the "small oscillations" approximation. A more general, but more abstract, "stationary state" is defined by virtue of the stability of the solutions of the Hamilton equations, just in this article recognized.

The global character of the model is assured when world data of variables are used. A very interesting feature of the model is that the path to a scenario of sustainability is given in terms *analogous* to the Newtonian Dynamics.

Keywords: unique dynamical system, Volterra generalized model, "conjugate" Hamiltonian pairs, quasi-periodic motions, Lyapunov stability, global stationary state.

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Introduction

Nicholas Georgescu-Rögen, who had previously tried to give thermodynamic laws to Economics to better take in account the consumption of natural resources (1971) [12a)], and his disciple Herman Daly proposed in the last Seventies a "Steady State" (1977) as a global objective for a sustainable economic scenario ([12b)] [13]), a kind of answer to avert the catastrophe forecasted by the well-known report "*A limit to growth*" (1972), in which "the predicament of mankind" was illustrated in terms of "bell" curves - the death of the global system when escapes control even *only one* of the model fundamental parameters, like population, food, energy and pollution.

The steady state proposal raised a decadal debate (see, e.g. [I]), interweaved with the development of theories and models coupling ecological and economic issues, which had started from Gordon's model of the impact of fisheries on the economics of open access (1954). The *bioeconomic modeling*, stemming from Gordon's model as well as from other inspirations, is characterized, mainly, by a dyadic conception in which the dynamics of coupled ecological-economic systems is interpreted in term of an *"interaction" between the two separated systems* (see, e.g. [II],[III],[IV],[V]).

Here, on the contrary, the authors propose a global "*stationary state*" following the generalized Volterra model, translated in the Hamiltonian formalism. This makes possible to "conjugate" every pair of variables, one economic, the other one ecological, in describing the time evolution not of two separated though interacting systems but as a *unique dynamical system*, as the two crises seem to strongly demand.

When the model is applied to the two pairs, N = 2, of conjugate variables - GDP/Total energy consumption and Complexity/CO_2 emissions - it is possible a suggestive geometry in its "phase space": *the trajectories of the entire system are curves wrapping a torus, whose set is the "stationary state" we were looking for*. Those trajectories are "*quasi-periodic motions*", characterized by *two frequencies* for the values of which a good estimate is given in the "small oscillations" approximation, as it is allowed by the reported data in Sect. 1.
The importance of the chosen variables is deeply rooted both in Economic and Ecological researches. The global character of the model is assured when the data of variables are taken from the available world data handbooks; the model obviously works also for world geo-economical subareas.

Beyond the constraints applied to the N = 2 model so that it can work, the stability of the solutions, here recognized for the Volterra generalized model (N > 2) in the Hamiltonian version, allows us to construct a more general "stationary state", albeit more abstract, richer than a "steady state" not only as a mathematical representation but also for an in principle wider choice of values available for the ecological-economic variables.

Last but not least, the model provides a tool according to which even the evolutions of an economic-ecological system over time can be described in a similar way to those typical of physics; and, mainly, a theoretical path moves towards a sustainable economy or, better, to an overall sustainability.

1. The double crisis and the need of a "global stationary state"

The upsetting consequences of the environmental crisis, mainly of the climate change, are struggling to become part of public knowledge and of the awareness of both the individuals and the human society (for a brief survey see [1], [2]). In some previous papers global data have been given, and their sources, showing the severity of the environmental crisis ([2]): from the land grabbing, even for the freshwater resources ([3 a), b)]), to the spreading of drought ([4]), to the degradation of coral reef ([5 a), b)]). And, above all, what has been called the greatest threat of this century, the climate change, or rather, the *already occurred transition* to climate instability ([6], [7a), b), c)]). On the other hand, the mainstream Economists seem deaf to the predicament of environment, despite that the climate instability and its dramatic consequences will last for the coming decades; no longer, then, an emergency, but a context inside which the economic policies should be evaluated. A remarkable exception has been, more than ten years ago, the *Stern Report* about severe and *global* damages on GDP due to climate change in a "business as usual" trend [8].

The current crisis of capitalism is a crisis of overproduction, whose peculiar quantitative feature, due to technological innovation in the global market, makes insurmountable the contradiction between the increase in supply and the market's ability to absorb it ([1], [9]). Are there new economic ideas to manage the two crises? "Weather forecasts" models have been recently proposed by the Institute for Economic Complexity (IEC), substantially as a branch of the Institute for New Economic Thinking (INET)[1] (see [10]). Those models introduce "Fitness" and "Complexity", two non-monetary and non-income based metrics, where Fitness is a measure of competitiveness of countries and Complexity is the level of sophistication of products ([11 a), b)]). The "*phase diagrams*", also "*phase portraits*", of the "weather forecasts" (see Fig. 1) strongly suggest the usual schemes and tools of the analysis of a two-dimensional nonlinear dynamical system; their constructions and definitions are very similar to those describing the fluid dynamics regimes ("laminar flow", "chaotic flow") with their abrupt changes, that is one of the main tool for modeling weather [2].

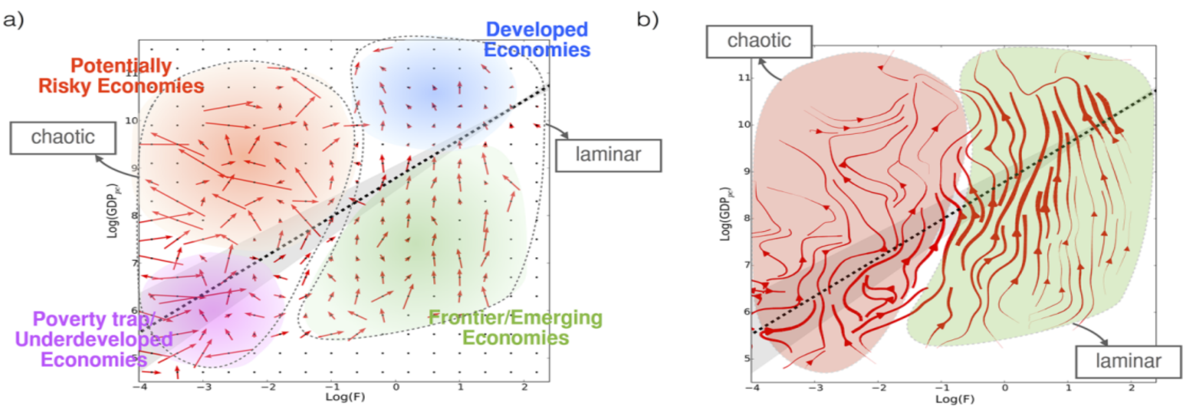


Fig. 1 "a) A finer coarse graining of the dynamics highlights two regimes for the dynamics of the evolution of countries in the fitness-income plane. There exists a laminar region in which fitness is the driving force of the growth and the only relevant economic variable in order to characterize the dynamics of countries. We argue that the evolution of countries in this region is highly predictable. There is also a second regime, which appears to be chaotic and characterized by a low level of predictability. In the laminar regime, we also find two different kinds of evolution patterns for the emergent countries and developed ones respectively; b) we report a continuous interpolation of the coarse grained dynamics to better illustrate the two regimes of predictability." [11 b)]

However, also the "weather forecasts" don't seem to provide a proper answer to the issue of a sustainable economy with respect to environmental crisis that is our problem; besides, those models fit only with the brief-medium term.

[1] A cultural non-profit economic research organization, founded in 2009 by George Soros, to build a global community of new economic thinkers to create new ideas towards our economic future. To this aim INET gathers from all countries hundreds of economists, among them several Nobel Prizes like Krugman, Stiglitz, Sen and so on, despite the questionability of his founder.

[2] "Physicists make 'weather forecasts' for economies" is the title of a news on Nature (23/2/2015).

In conclusion, *all economic models* [9], also those more critical and up to date we have above briefly summarized, *keep separated the economy from the ecology*. On the contrary, we are firmly convinced that a global "*stationary* state" model requests, in the perspective of a global sustainability, the study of the dynamics of an *only one ecological-economic system*, as Georgescu-Rögen and Daly did, but with the scientific tools to which we have just referred in the title.
"Global", as regards its applicability and the scenario that it would represent, the model we'll introduce is "local" from the point of view of the small values allowed to the oscillations of variables; in other words, our reasoning is typical of the analysis of *small oscillations in a neighborhood of a stable equilibrium.*

How realistic is the reduction to small oscillations with respect to the expected trends? We immediately provide the figures that support this reduction: a 2018 forecast by the International Monetary Fund predicts that up to 2023 world GDP rates, especially for advanced countries, will be half of those pre-crisis; referring to the data reported in *Key World Energy Statistics* 2018 by the International Energy Agency (IEA), the Agency of the OECD countries, it is easy to calculate that in the period 2011-2016 the growth rate of CO_2 emissions was 0.0062/year, that is about a quarter of the growth rate in 2006 - 2011 (0,0233 / year).
This behavior of two fundamental parameters, which will be used in the "4-variables first attempt" in Sect. 3, seems to legitimize resorting to the framework of the small oscillations.
However, in the Appendix of this article we will give a definition and an analysis of the "stationary state" in a more general context than the one referring to the regime of small oscillations.

2. Lotka-Volterra and Goodwin's models: Hamiltonian systems

The description of an economic trend in terms of a nonlinear two-dimensional dynamical system was performed, fifty years ago, by Richard M. Goodwin, when he applied the Lotka-Volterra model to the Economy, illustrating what has been defined as the "class struggle" model ([14]).
The cycle "predator – prey" of two populations in competition, that Alfred J. Lotka [15] and Vito Volterra [16] had proposed in the Twenties, has provided to the Applied Sciences – Populations Dynamics, Biology, Biophysics, Health Epidemics, Chemistry and, in particular, Economy – a model of oscillations between a minimum and a maximum, compatible with the existence along all the cycle of the two variables defining the model, because the minima are always greater than zero: a well-known model, incredibly largely applied, of a dynamical system in *two dimensions*, as many as are the variables.
Just looking at the cycles of the model in its "phase portrait", we'll sketch the basis for ***an "ecological-economic" model of a global stationary state*** towards a sustainable economy. Several of the arguments hereafter exposed have been already raised in [9].

"Cycle" has its definition there where it started, in Mathematics or Physics: a *closed curve that describes the trajectory over time of a system that starting from an initial state to it comes back in a finite time.* The behavior in time of the variables which generate a cycle is given by *periodic functions*, like sine or cosine, whose characteristic is the *constant value*, along *all the time evolution*, of the oscillation *period* and of the *amplitude*, minima and maxima.
The oscillations in time between the maxima and minima of the variables studied by Economics can have a periodic behavior in time, in general, *only for a limited time interval*; for a longer time, the oscillation parameters do not remain constant but *vary largely*, thus losing their periodic characteristics. Then, no *return of the system to its initial state and no cycle*, but, if any, an open

curve, typically a spiral, along which there is a continuous growth, loaded of social and environmental contradictions.

In the Lotka-Volterra model the two variables are the number of individuals of the two species – predator and prey – that varies in time, *in first approximation*, as a sine or cosine (see Fig. 2). The *phase displacement of the two curves* is easily understood because *one species eats the other one.*

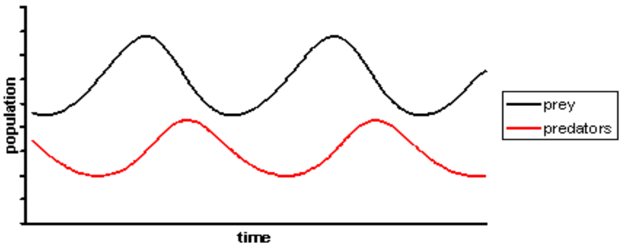


Fig. 2 Behavior in time of numbers of prey and predators

Correspondingly to that time behavior the phase diagram is constituted by cycles, so many how many are the possible initial conditions for the evolution; the phase diagram plane is the *system phase space*. In this space every point has the role of a well determined state; thus, *the phase diagram gives a geometric vision of the orbits of the system as a sequence of all states along which the system evolves in time* (see Fig. 3).

Also in Economics the existence of cycles in the proper sense of the term is contemplated, when, as in the Goodwin's model, the number of "predators" and "preys" is replaced, respectively, by the "share of the product of the worker", a variable linked to the wage rate, and the "employment rate" [14]. The employed workers have the role of predators, because the wages reduce profits and hence investments and this turns in an increase of unemployment; by this reason the model is also known as "Goodwin's class-struggle model".

The phase portrait of the Lotka-Volterra and Goodwin's models is represented in Fig. 3; only in the first quadrant because of the sense of the two variables, that are necessarily positive ($q > 0$, $p > 0$).

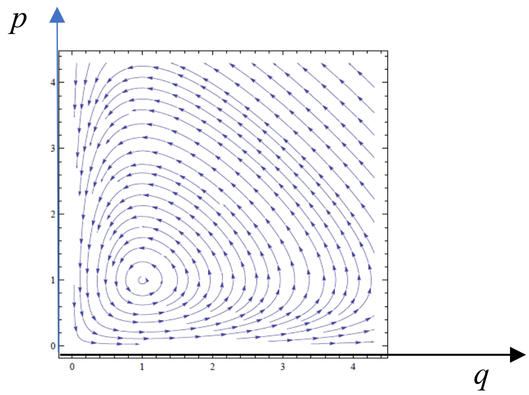


Fig. 3 Phase portrait of the predator-prey type model (drawn from [17])

The orbits exhibit an *elliptical behavior* only near to the equilibrium, the point around which all curves turn. The construction of the orbits in Fig. 3 highlights how it is possible to do, starting from the vector field defined by the second members of differential equations underlying the model.

The Lotka-Volterra and Goodwin models, and their cycles, represent typical "*stationary*" models: nothing is "steady", i.e. static, but the evolution of the system along a cycle – oscillations over time between minima and maxima for both the variables – repeats itself with the same characteristics in any period. Also the revolution of the Earth around the Sun is, in first but good approximation, a stationary phenomenon.

It's worth to remark that Goodwin's model is the first that tries to combine cyclical behavior and economic growth. The cycles ultimately depend on the fact that *the Goodwin's model is based on differential equations of evolution in time* of Lotka-Volterra type. This is not usual in economic theories, as well as in other evolutionary sciences, because of the impossibility of defining a "dynamic" for the system as instead is possible in the powerful Newtonian scheme (the "force" is the *cause* of variation of the motion) thanks to the simplicity of phenomena that it schematizes. Indeed, what should play the role of the "force" for an economic or also an environmental system?

Finally, the success of Lotka-Volterra model relies on the ***Hamiltonian character*** of that nonlinear two-dimensional dynamical system. A dynamical system is Hamiltonian if a function exists, the Hamiltonian function *H*, constant over time and that summarizes the information on the "energy" of the system, also when it has no more the meaning of energy as defined in Physics.
The area enclosed by the cycle provides a numerical value connected to the energy, which grows with the length of the orbit; therefore if you run the same cycle N times, that is, you multiply its length by N, the required energy will correspondingly increase. If the orbit were not closed but were spiraling outwards as in a model of unlimited growth, the required energy would tend obviously to infinity.

Since all the natural resources that supply energy to the system - fossil sources, minerals, raw materials, soil, water, biomass - are limited except for solar energy, a sustainable ecological-economic model must satisfy the known constraint whereby in the cycle the speed of consumption of each natural resource is lower than the speed of its reproduction, or takes into account the availability of that resource to plan its consumption and give time to science and technology to find sustainable substitutes.

The *generalized Volterra model* is the one extended to $N \geq 2$ *pairs of variables*, and the number N is called "*degree of freedom*" of the system [18]. For our "stationary state" model we'll use the generalized Volterra model transcribed in the Hamiltonian formalism [19].

A state of the system is a set of 2N variables $(q_1, .., q_N; p_1, .., p_N)$, a point in the 2N-dimensional phase space; but *in order the system to be defined hamiltonian*, in each pair (q_i, p_i) with i = 1, .., N, the variable p_i must be "***conjugated***" to q_i by means of the function $H(q_1, .., q_N; p_1, .., p_N)$.
This "*conjugation*" involves the Hamilton equations, whose general form is reported in Appendix, and is the translation into the Hamiltonian formalism of the fact that each pair (q_i, p_i) consists of "***one who eats the other***". This is the crux of the Hamiltonian character of the equations underlying the Lotka-Volterra, Goodwin and generalized Volterra models $(N \geq 2)$

This opens up the possibility of constructing a stationary state of an economic-ecological model by coupling an economic variable to an ecological one ***in a unique dynamical system***, provided that in each pair there is "one that eats the other". As we shall see, there are quite easily good candidates.

The global character can then be ensured by using the global values provided for the model parameters by the dedicated data manuals; when using the values available for each geo-economic region – advanced countries, developing countries – or for a sub-region, the model will go on running in the same way.

A final observation on the insistence on the term "*stationary state*" instead of the more usual "steady state". The mathematical meaning of "steady" is "fixed"; and such are the equilibrium states of the

differential equations of the Hamiltonian system at the base of the model. Thinking of the phase diagram, the "steady states" are a set of points, a set that is too "poor" to represent a global ecological-economic state. The *stationary state* can instead have a geometrically richer representation, if one manages to construct it as a set of orbits, such as, for example, the set of ellipses "around" of the equilibrium point of Fig. 3.
The Hamiltonian "technicalities" applied to the generalized Volterra model ($N \geq 2$) and to the following "first attempt" are given in Appendix.

3. A first attempt with four variables (N = 2)

The simplest case, after the one studied by Lotka-Volterra and Goodwin, N = 1, is that of identifying two pairs of variables, N = 2, which both enjoy the Hamiltonian "conjugation", i.e., that exhibit a behavior of the predator-prey type. Good candidates to start with are the pairs: "Final energy consumption / GDP" and "CO2 emissions / Complexity".

Why precisely these four variables? Apart from their undoubted importance in economics and environmental studies, the most specific aspect is that each of the two proposed pairs is of the predator-prey type. Energy consumption has not always been indicated as the "engine" of the economy? Therefore, for its growth the GDP "***eats***" energy. Similarly, low carbon emissions correspond to an economic system capable of producing more sophisticated goods, that is, of higher "complexity" (the old, less sophisticated economies have higher carbon emissions). Thus, complexity "***eats***" CO_2 emissions.

We will assume that the Hamiltonian function of the model be constructed so as to enjoy the "***separability***", a general property here applied in the case N = 2; it is not a bizarre hypothesis because it is *the condition, even if only necessary*, *for the "integrability*"of the differential equations at the base of the model, i.e. of its solution. Simply put, the description of the system, which is very complex, is simplified in many separate phase planes, two in our case, in each of which there is a phase diagram of the type shown in Fig. 3, that is, in each l-plane, $l = 1, 2$, p_l is a function only of q_l.
However, between the several planes a sort of coupling remains, due to which the phase diagram of a plane may be affected by the diagram of another plane. We will then request, in addition to separability, the ***decoupling*** between the two planes we are dealing with: *the oscillations over time (see Fig. 2) that generate the cycles of the phase diagram of the pair* (q_1, p_1) *must be completely independent of those that generate the cycles in the plane* (q_2, p_2).

This decoupling, required for the two pairs of "crossed" variables: "CO_2 emissions / final energy consumption" and "Complexity/GDP", would be false, for the first pair, if based on the historical trend of the two variables, but is becoming more and more realistic if we look at the sharp decrease in the growth rate of CO_2 emissions in the last twenty years, previously reminded (Sect. 1) and the growing role of carbon-free energy sources. The correlation between Complexity and GDP is more complicated because a higher Complexity could promote GDP, but, in fact, the most creative and "intangible" assets are excluded from the composition of GDP.
Following the above assumptions, we can combine the direct sum of two non-interacting predator-prey models; thus, the Hamiltonian H_0 of the *4-variables economic-ecological model* is given by *the sum of two independent Hamiltonians* H_{01} and H_{02}: at this point, our model is mathematically composed by two copies of the same system.

Remembering the role of Predator/Prey pairs, respectively GDP/CO_2 emissions and Complexity/Energy consumptions, one has

$$H_0 = H_0\,(q_1, q_2; p_1, p_2) = H_{01} + H_{02} =$$

$$= \varepsilon_1 q_1 + \eta_1 p_1 - a_{21}\, e^{q_1} - a_{12}\, e^{p_1} + \varepsilon_2\, q_2 + \eta_2 p_2 - a_{43}\, e^{q_2} - a_{34}\, e^{p_2},$$

where, now:

q_1 = world energy consumptions; p_1 = world GDP; q_2 = CO_2 world emissions; p_2 = world Complexity; ε_1 = world GDP growth rate; η_1 = world energy consumptions growth rate; ε_2 = world Complexity growth rate; η_2 = CO_2 world emissions growth rate; $a_{12} = -\, a_{21}$ is the interaction coefficient between energy and GDP; $a_{34} = -\, a_{43}$ is the interaction coefficient between CO_2 emissions and Complexity.

Each H_{0l}, $l = 1, 2$, is *separable* because, as it is evident, in each phase plane p_l is a function only of q_l and E_l, where E_l is the "energy" in the sense we have illustrated in sec.1; then H_0 *is separable.*
Such system has an equilibrium configuration corresponding to the point P in R_4, where R_4 is the ordinary vector space in four dimensions:

$$P = ((\log (\varepsilon_1/a_{21}), \log (\eta_1/a_{12}); \log (\varepsilon_2/a_{34}), \log (\eta_2/a_{43})),$$

in which each ratio, that appears as argument of the log function, has to be greater than one. Under this condition P belongs to the first octant; it is a "center", as it is usual for equilibrium points of Hamiltonian systems.

Each of the phase diagrams belonging to an l-plane, $l = 1,2$, consists of an infinite number of cycles, and, among them, a *cycle is obtained by choosing a pair of initial data*, i.e., the initial state of the system. Determining the initial state - a point in the phase diagram - selects the cycle passing through that point because for each point only one cycle can pass (uniqueness theorem); therefore *two cycles* can be selected in the respective l-planes, C_1 and C_2, such as those represented in Fig. 4 a), b).

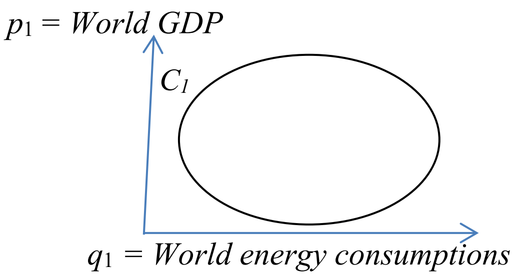


Fig. 4a)

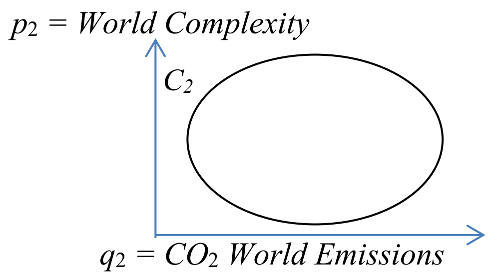
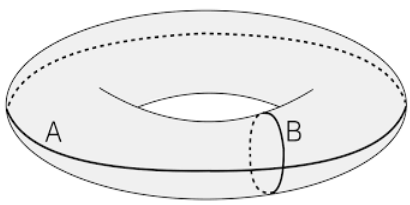


Fig. 4b)

The cycles C_1 and C_2 are represented as ellipses; this representation is "faithful" in a "local" view, that is when we consider small oscillations around a stable equilibrium point (see Fig. 3).

This four variables model is then susceptible of a suggestive geometric sketch: the *topological product* of the cycle C_1 in the (q_1,p_1)-plane for the cycle C_2 in the (q_2,p_2)-plane gives rise to a torus, a surface like a ring-shaped donut (see Fig. 5 a), where there is also an indication on how to do the "product"); and the evolution of the system would be represented by a curve that winds the torus, the trajectory (see Fig. 5 b)).



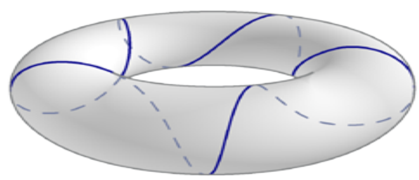


Fig. 5 a) a torus as a topological product of two ellipses b) a trajectory on the torus

Further, each cycle C_l is a level line determined by the equation $H_{0l} = E_l$ and is therefore parametrized by E_l; geometrically speaking, C_l is the "foliation" of a two dimensional torus (see Fig. 5 a)) like if we were cutting the latter with the l-phase plane.

A more realistic model for a stationary state could request more pairs of variables than two. In this case the representation of the evolution loses the geometric character of visibility, so useful in the previous example. Then, it is reasonable to substitute the complex and no more representable figures as are those determined by the topological products of more than two cycles, with the corresponding analytic expressions: the so-called *"quasi-periodic motions"*, whose expression will be provided in Appendix.

4. The "stationary state"

The set of trajectories that wrap around the "donut" - the "Hamiltonian flows" on the torus - constitute the "stationary state" we were looking for, certainly mathematically richer and more versatile than the "steady states" advocated in the past. Each trajectory is what we have called "quasi-periodic motion" on the torus; it is animated by two frequencies, that is, as many as there are degrees of freedom of the system (N = 2). The problem is now to calculate those frequencies, or to have a good approximate value, so as to achieve a resolution of the model.

Then, the Hamiltonian formulation of the generalized Volterra model (N ≥ 2) provides the possibility of reaching a quantification when there are N constants of the "motion": the ***actions*** J_k, each defined, ***by virtue of separability***, in the k-plane of the pair (q_k, p_k) as integral along the cycle C_k:

$$J_k = \int_{C_k} p_k\,(q_k, \mathrm{E}_k)\; dq_k \qquad k = 1, .., \mathrm{N}; \qquad E_k \text{ is the "energy" associated to the cycle } C_k,$$

(the numeric value of J_k is the area enclosed by C_k).

If those N constants "commute" between them, in the sense required by the Hamiltonian formalism, there is the ***complete integrability*** of the system (see Appendix), that is the complete *knowledge of the solutions of the differential equations at the base of the model*: $\boldsymbol{q_i = q_i(t), p_i = p_i(t)}$, $i = 1, .., \mathrm{N}$. This happens, for example, if an algebraic condition on the coefficients ruling the interaction among the different species is imposed [19], *but it is too abstract with respect to its applicability in terms of our model* [20].

Anyway, the Hamiltonian theory supplies the ***frequencies*** of the quasi-periodic motion for N ≥ 2 through the (N x N) matrix, whose elements are the derivatives of the "energy" with respect to the "actions": $\frac{\partial}{\partial J_k} E_l$, $(l, k = 1, .., \mathrm{N})$, where E_l is the "energy" of the cycle C_l.

For N = 2, the decoupling hypothesis implies that the square matrix (2 x 2) of the frequencies is reduced to a "diagonal" form, i.e., a simpler form for their determination (the derivatives with "crossed" indices are null). A simplification that leaves us with the expression of two elliptical integrals very hard to calculate (see Appendix), but the adoption of the "local" point of view ("small oscillations") comes to the aid allowing to obtain an estimate of good approximation for the two frequencies:

$$\nu_1 = \sqrt{\varepsilon_1 \eta_1}\,, \qquad \nu_2 = \sqrt{\varepsilon_2 \eta_2}\,,$$

that enables us to get the "quasi-periodic motion" of the system over time (see Appendix).

In conclusion, each of the trajectories on the "donut" is a "*quasi-periodic motion*" on the torus, whose two frequencies can be approximated, or submitted to a computing program starting from the

complete expression given in Appendix. Varying from one trajectory to another depends only on the initial state chosen, but the parameters defining ν_1 and ν_2 remain the same.
All this is true *if the two frequencies are real numbers between them incommensurable*, while if their ratio were a rational number we would have the reduction from a quasi-periodic motion to a simply periodic motion; but this is as "unlikely" as finding a rational number as a result of the relationship between two real numbers.

The "richness" that we want to attribute to the representation of a stationary state holds more generally, that is *without any request for separability or decoupling*, but in a decidedly more abstract context, by virtue of the theory of stability. We refer here to the classical *Lyapunov stability*, that is the stability with respect to the small perturbations on the initial data*: the solution of a system of ordinary differential equations is stable if any other solution, which starts from an initial state "close" to that of the stable solution, goes on keeping itself as close as you want to the stable solution* ***for all time*** [3]. [21 a), b), c)]
All solutions which start "close" to the stable ones are usually called "perturbed".

The Lyapunov stability responds to a general request in every scientific research on the phenomena of evolution over time: we want to ensure that if the initial state of the system is not precisely known, or if we made some small mistakes in preparing it, this fact does not affects all the evolution of the system. In other words, if the difference due to an error in the determination of the initial state is small, we would like that the distance between unperturbed trajectory and the perturbed one remains small for all time, otherwise the evolution would be unstable.

Stability is therefore *a property of control over the spatial evolution of a phenomenon for each duration*. In fact, all the perturbed solutions are trajectories which, by definition of stability, are found within an arbitrarily thin "*tube*" formed by them, all the time ($t \rightarrow \infty$), around the trajectory of the stable solution.
Volterra has shown that the solutions of the differential equations of the generalized Volterra model are stable [19]; on the other hand, the stability continues to hold even if we "transcribe" that model in the Hamiltonian formalism, a not previously known result [22].

Then, *the stationary state of the Hamiltonian model based on an unique ecological-economic dynamical system with* $N \geq 2$ *"degrees of freedom", without any separability or decoupling request, is represented by the set of all those perturbed trajectories, the just before mentioned tube*, *when the proper ecological-economic parameters are adopted*.
Mathematically, this stationary state is a compact and invariant "manifold", as it easy to show, moreover structured by infinite "sheets" of "energy", to each of which a perturbed solution belongs (each "sheet" is determined by the initial data (q°_i, p°_i)) (see Appendix).
Economists will not jump for joy, but the determination of such a manifold is a topos of the theory of dynamical systems; and provides an object a slightly less vague than a *steady state*.

Conclusions

A subsequent step towards a more realistic model can be to remove the decoupling conditions. This would imply to add some proper term to the Hamilton equations of the model, but also a slight linear modification to the second members yields a destruction of the closed orbits as the model at the base is not robust. Thus, one could question if there exist "Hamiltonian perturbations" to alter the

[3] Lyapunov stability is a topological property that would coincide with ***continuity*** if it were requested only for a finite interval of time and not *for all time*.

Hamiltonian function in such a manner that the phase portrait be changed without losing the closed curves which are the essential feature of the model.

The general answer to this problem can be found in the celebrated Kolmogorov-Arnold-Moser (KAM) theory; but at the present it could seem a bit to shoot a fly by a gun. Anyway, results for a number of pairs N(even) ≥ 2 could be obtained if the Graff's theorem [23] were applicable to the model Hamiltonian system, included the rigorously approximated values guaranteed for the frequencies of the existing quasi-periodic solutions. It is a different point of view from that we have followed, mainly referred to the Hamiltonian version of the generalized Volterra's model; a focal problem is to verify if, in the case $N \geq 2$, the Hamiltonian function is likely to satisfy the conditions for which the Graff's theorem holds. It is a suggestion for further steps of our job.

We know well that priorities and problems concern something else, however there is an important aspect that legitimizes such an "abstract" space for reflection: researchers must play their role, on pain of being responsible, on their part, for the consequences for the whole humanity. As the surprising failure of the economic mainstream has taught, whose theories and models, followed or justified by the governments of the most important countries of the world, have not been able to foresee the current global economic crisis. Similarly, apart from isolated exceptions, it was serious to ignore the environmental crisis and the consequent urgency of the "spaceship earth" economy proposed by Kenneth Boulding more than 50 years ago or, a decade later, the need to reach a global "steady state", as claimed by Georgescu-Rögen and Daly. There is therefore space and need to emphasize that other conceptions, other scientific instruments and models are possible, different from the dominant ones that have proved to be so seriously inadequate.

Appendix

In his celebrated book about *the struggle for life* [19], Volterra introduced a more general model in order to describe the competition of *n* biological species that, two by two, eat each other. Then, the system of differential equations to represent this competition can be written

1) $$\dot{x}_i = \varepsilon_i\, x_i + \frac{1}{\beta_i} \sum_j a_{ij}\, x_i\, x_j \qquad (i,j = 1, 2, \ldots, n) \ ;$$

where x_i is the number of individuals of the species i ($x_i > 0$), the ε_i's are the natural growth coefficients, a_{ij}'s are the interaction coefficients linked to the probability of encounters between the individuals of two species i, j. It's worth noting that if all ε_i's have the same sign it's impossible both a stationary solution and small fluctuations. With the change of variables $x_i \rightarrow x_i/\beta_i$, one can assume $\beta_i = 1$. The differential equations system 1) is the *generalized Volterra model* because is a generalization of the case $n = 2$ ([18], [19]). Following [19], we define the *quantity of life* q_i, i.e., how many individuals of the species *i* are still alive, step by step in time, till an instant *t*:

2) $$q_i = \int_0^t x_i\ (\tau)\ d\tau\,, \qquad x_i\,(0) = 0 \qquad (i = 1, .., n)\,,$$

that allows to rewrite the system 1) as a system of second order differential equations

1') $$\ddot{q}_i = \varepsilon_i\, \dot{q}_i + \Sigma_j\, a_{ij}\, \dot{q}_i\, \dot{q}_j \qquad (i, j = 1, .., n).$$

Then, the function $$H' = \textstyle\sum_i \left(\varepsilon_i q_i - \dot{q}_i\right) \qquad (i = 1, .., n)$$

is a *first integral* of 1'). If now we introduce the variables

$$p_i = \log \dot{q}_i - \tfrac{1}{2}\, \Sigma_j\, a_{ij}\, q_j \qquad (i, j = 1, .., n),$$

the function H' becomes

3) $$H\,(q_1, .., q_n; p_1, .., p_n) \;=\; \sum_i \varepsilon_i q_i - \sum_i e^{\,(p_i + 1/2\, \Sigma_j\, a_{ij}\, q_j)} \qquad (i = 1, .., n)$$

and the corresponding Hamilton equations are:

4) $$\dot{q}_i \;=\; \frac{\partial}{\partial p_i} H \qquad \dot{p}_i \;=\; -\frac{\partial}{\partial q_i} H \qquad (i = 1, .., n)\,,$$

where now H is given by 3) and is a *first integral* of 4), that is $\frac{d}{dt} H = 0$ as one can directly verify[1]

The Hamilton equations 4) make also explicit *what* has to be meant for ***conjugacy*** relationship, that each pair of Hamiltonian variables (q_i, p_i) *must satisfy*, synthetized in the association

$$\dot{q}_i \longrightarrow \frac{\partial}{\partial p_i}\;;\quad \dot{p}_i \longrightarrow -\frac{\partial}{\partial q_i}\,.$$

In view of finding ***separability*** for the Hamiltonian 3), i.e., that every p_i can be expressed as a function only of its conjugate q_i and some constants, one can show the existence of the following n first integrals of the Hamiltonian system 4), time dependent but functionally independent:

5) $$\varphi_j\,(q, p, t) = p_j + \tfrac{1}{2} \sum_i a_{ij}\, q_i + \varepsilon_j\, t \qquad (i = 1, .., n;\; j = 1, .., n)\,^{2}.$$

We can easily eliminate time t from 5), but the final expressions we obtain for each φ_j (q, p) –linear in the single p_j and in all q_i's – do not lead to the separability, such as we have previously requested. Another bad news is that the φ_j's don't "commute" [3] and, by this reason, our system is not *completely integrable* [3], i.e., it cannot enjoy all rich properties that compete to such systems, included the *existence of quasi-periodic solutions*, fundamental for our model.

At this point it is correct to notice that in his paper Volterra gives *an algebraic characterization for a completely integrable evolution of the system generated by* 3), essentially through a simple linear

[1] $\frac{d}{dt} H = 0$ is a general feature of the Hamiltonian formalism, whatever be the analytic expression of H. In fact: $\frac{d}{dt} H = \sum_i \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i\right) = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$, $i = 1, .., n$, where the substitution in the last equality is possible *only because* q_i and p_i *are thought as solutions of* 4), even if not explicitly known; the result is zero, thus verifying that H is a constant of motion. This operation explains well the meaning of "derivative along the motion".

[2] The derivative with respect to time of each φ_j is performed in this way

$$\dot{\varphi}_j\,(q, p, t) = \sum_i \left(\frac{\partial \varphi_j}{\partial q_i} \dot{q}_i + \frac{\partial \varphi_j}{\partial p_i} \dot{p}_i\right) + \frac{\partial \varphi_j}{\partial t} = \sum_i \left(\frac{\partial \varphi_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \varphi_j}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial \varphi_j}{\partial t} =$$
$$= \sum_i \left\{\left(-\tfrac{1}{2}\, a_{ij}\, e^{(p_i + \frac{1}{2}\Sigma_k\, a_{ik}\, q_k)}\right) - \left(\delta_{ij}\varepsilon_j - \tfrac{1}{2}\, a_{ij}\, e^{(p_i + \frac{1}{2}\Sigma_k\, a_{ik}\, q_k)}\right)\right\} + \varepsilon_j = 0,$$

where $k, i = 1, .., n$; $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Let's remark, even here, that in the second equality we can do that substitution of $\dot{q}_i$ and $\dot{p}_i$ only because we are performing a "derivative along the motion".

[3] Given two functions f, g depending on the 2n variables (q_i, p_i) in a $2n$ phase space, one defines the Poisson bracket between them: $\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$, $i = 1, .., n$. If $\{f, g\} = 0$, i.e., if f and g ***commute***, they are said to be in ***involution***. If the Hamiltonian system admits n constants of motion and *they are in mutual involution*, the system is ***completely integrable***, that is one can obtain by analytical calculations the solutions of the Hamilton equations. In our case it results $\{\varphi_i, \varphi_j\} = a_{ij}$, thus no involution, no complete integrability.

transformation of the first integrals 5) ([19], see part II, sect. 5); but this kind of approach *makes problematic the application of that condition for complete integrability to an economical-ecological context written in the Hamiltonian formalism* [20]. Thus, we have to abandon, even if grudgingly, the rich cornucopia provided by the complete integrability of the generalized Volterra model; and in the lack of that exploitation, *we are obliged to do many assumptions, the fundamental of which is the existence of a closed curve C_j* (like a cycle of Fig. 3) in each phase plane (q_j , p_j), associated to the oscillations that the variables q_j , p_j run, independently, in time. The existence of such a cycle *is not easy* in general to demonstrate, unless one assumes *not only the separability of the Hamiltonian function but also its decoupling*, as we have done in Sect. 3. When a cycle exists in each phase plane (q_j, p_j), it can be shown through several steps *implying only the separability and not a decoupling hypothesis* [24], that the final expression for the q_i's is that of a *quasi-periodic motion*:

6) $$q_i\,(\mathrm{t}) = S_i\,(\nu_1\mathrm{t} + \varphi_1, \ldots , \nu_n\mathrm{t} + \varphi_n) \qquad (i = 1, \ldots , n)\,,$$

where the S_i's are *periodic functions of each argument*, of period 1, a "multi-periodicity"; and all frequencies $\{\nu_1,..,\nu_n\}$ are given by the derivative of the energy E with respect to the "actions" J_j:

7) $$\nu_j = \frac{\partial}{\partial J_j}E \qquad (j = 1, \ldots , n)\,.$$

J_j is the "*action*", so called because in Mechanics has the physical dimension of the product of an energy for a time, that we already defined in Sect. 4 and report here for the reader's convenience:

8) $$J_j = \int_{C_j} p_j\,\left(q_j, c_1, \ldots , c_n\right)\,dq_j \qquad (j = 1, \ldots , n)\,.$$

The functions S_i express the *quasi-periodic character* of the "motion", also in the case where a decoupling condition is assumed, and the way we have followed highlights, without specialized further considerations, the *intrinsic character of the "quasi-periodic" motions to the structure of Hamiltonian function* [24]. Moreover, by that way, one doesn't make the mistake to which the students which follow the traditional method are induced – and that seems to appear stealthily also in some renowned university book – to think that the frequency ν_j given in 7) is the frequency of q_j. *On the contrary, it is not true, in general, that the q_i's are periodic functions of time*.

About frequencies of our model (N = 2), there is to observe that their determination requests a bit of caution because, *if the two systems were coupled* frequencies should be given by a not diagonal matrix whose four elements are the derivatives (see Sect. 4): $\partial E_l/\partial J_k$, $l, k = 1, 2$ [20]. Luckily, the hypothesis of *reciprocal independence between the two systems* of our model reduces that matrix to a diagonal one, off-diagonal elements being zero, each diagonal element being

$$\partial E_l/\partial J_l, \qquad l = 1, 2.$$

Because it appears easier to calculate $\partial J_l/\partial E_l$ and after to take the inverse, as it is permitted by diagonality, one obtains *in a neighborhood of an equilibrium*

7') $$\nu_l = \frac{\partial E_l}{\partial J_l} = 1/\left(\frac{\partial J_l}{\partial E_l}\right) = \left\{2\int_{q_{l,min}}^{q_{l,max}} \frac{b_l\,dq_l}{[-1+\exp\left(f^{-1}(E_l - a_l\,f_l\,(q_l))/b_l\right)]}\right\}, \quad l = 1,2,$$

where $q_{l,min}$ and $q_{l,max}$ are the two intersections of the ellipse with q_l axis, as it is usual when dealing with a periodic motion whose phase portrait is an ellipse (see Fig. 6).

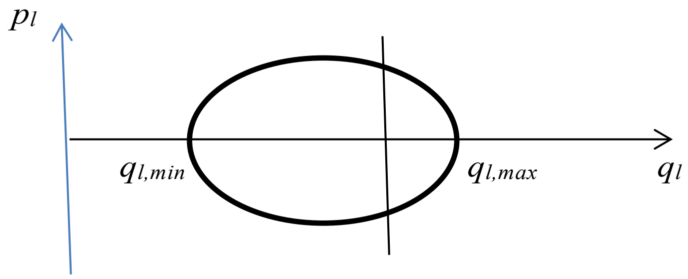


Fig. 6 If a point turns a complete round on the ellipse here above, the projection of the point on the axis q will double the path; therefore the time required, the period, is twice the time to go from $q_{l,min}$ to $q_{l,max}$ (the factor 2 in 7')). The straight line recalls the caution to be taken because the graph of an ellipse is not that of an invertible function.

For the sake of generality, *for an evolution not limited to small oscillations* the two required frequencies are:

7")
$$\nu_l = \frac{\partial E_l}{\partial J_l} = \oint \frac{b_l dq_l}{[-1+\exp(f^{-1}(E_l - a_l f_l(q_l))/b_l)]}, \quad l = 1,2,$$

where integration is extended over the whole cycle C_l, that in general is not an ellipsis (see Fig. 3). In 7') and 7") the coefficients a_l, b_l depend on the parameters of H_{0l}.

In front of the cumbersome calculations implied by the integration 7') or 7") – for a good review about the methods, see [25] – an approximation is available, that seems very suitable for an ecological-economic model whose solutions are small oscillations in time (see Sect. 1). Then, a good estimate for the two frequencies 7') can be found, by simple calculations, for each of them in its *l-plane* as the oscillation frequency associated to the orbit near the equilibrium (the projection of the point P in the *l-plane*). This well-known procedure demands a linearization of the Hamilton equations near P; thus, one obtains

9)
$$\nu_1 = \sqrt{\varepsilon_1 \eta_1}, \quad \nu_2 = \sqrt{\varepsilon_2 \eta_2} .$$

About stability

Volterra had already demonstrated Lyapunov stability for his generalized model, but only *in the natural coordinates* [18]. The consequences of giving a Hamiltonian form to equations system 1) have been explored also in [26], without giving the demonstration of existence and stability, that would involve a too complex matter (see [27], [28]).

In front of those difficulties it is a bit strange that none has before resorted to the powerful simplicity of the Lyapunov's method. In fact, one of its theorems [21 b)], *applied to our case*, provides the *stability of the unperturbed solution of 4) in presence of first integrals* like those given by 5), a conditioned stability, i.e., only in R_n [22]; but under a suitable form for those first integrals, it can be shown that a tout-court stability holds in R_{2n}.

Further, since their stability has been proved – stability holds also for a state consisting by emptiness – then the equilibrium points of 4) exist, because *the property of stability implies for a solution its existence in the future*. A good performance, if one thinks of the laboriousness of such a demonstration in natural coordinates ([18], [19]).

Our stationary state, *the set S of all trajectories corresponding to perturbed solutions* is *bounded*, i.e., it can be contained in a finite volume of R_{2n}, and is *closed*, i.e., it contains also the elements of its border, always by virtue of stability. Now, let's denote by $\mathbf{M} = S \cup \partial S$, the union of S and its border; then, because in each finite dimensional vector space R_N a set is defined "compact" if it is bounded and is closed, then $\mathbf{M} \in R_{2n}$ is *compact*.

Further, the set **M** is an *invariant compact set of R_{2n}* (an invariant compact manifold of R_{2n}). The *invariance* in time is due to the same character of the set **M**, *whose elements are solutions of equations* 4): if the state $(\boldsymbol{q}(t_0), \boldsymbol{p}(t_0))$ belongs to **M** at an arbitrary instant t_0, the solution $(\boldsymbol{q}(t, t_0), \boldsymbol{p}(t, t_0))$ that starts from $(\boldsymbol{q}(t_0), \boldsymbol{p}(t_0))$ will belong, by definition, to **M** for any $t > t_0$; this property is called *positive invariance*, that for autonomous systems, i.e., not explicitly depending on time, can be extended also to all $t < t_0$, that is, the *negative invariance*, thus realizing the *invariance* for any time.

We can add that **M** is not the poor and twisted set that it could seem, nor a destructured object like an ameba, because any perturbed solution belongs to an unique value of "energy" E; thus, to **M** can be uniquely associated a variety of infinite layers, where each of them is a surface level E, $0 < E \leq \bar{E}$ and $\bar{E}$ is the maximum value that E takes when x_0 varies in the closure of the ball containing all perturbed initial data.

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